NOTE THAT THE FIRST VALUE OF n IN A PROOF BY INDUCTION ON n COULD BE ANYTHING! (THIS ONLY CHANGES OUR FIRST STEP AND THE $\forall$ IN OUR SECOND STEP.)

1. TO PROVE $\forall n \geq 0$, $\underbrace{1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}(2n^3+3n^2+n)}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: LHS OF $P(0) = 0$ (NO TERMS TO ADD!), AND
RHS OF $P(0) = \frac{1}{6}(2\cdot 0^3 + 3\cdot 0^2 + 0) = 0$, SO LHS = RHS ✓

CLAIM: $\forall n \geq 0$, $P(n) \Rightarrow P(n+1)$ → I.E., $\underbrace{1^2+2^2+3^2+\cdots+n^2+(n+1)^2}_{\text{(LHS)}} \overset{?}{=} \underbrace{\frac{1}{6}\left(2[n+1]^3+3[n+1]^2+[n+1]\right)}_{\text{(RHS)}}$

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}(2n^3+3n^2+n)$ $(*)$

THEN $\underbrace{(1^2+2^2+3^2+\cdots+n^2)+(n+1)^2}_{\text{(LHS)}} = \frac{1}{6}(2n^3+3n^2+n) + (n+1)^2$ BY $(*)$

$= \frac{1}{6}\left(2n^3+3n^2+n + 6[n^2+2n+1]\right) = \frac{1}{6}\left(2n^3+9n^2+13n+6\right)$

AND $\underbrace{\frac{1}{6}\left(2[n+1]^3+3[n+1]^2+[n+1]\right)}_{\text{(RHS)}} = \frac{1}{6}\left(2[n^3+3n^2+3n+1]+3[n^2+2n+1]+n+1\right)$

$= \frac{1}{6}\left(2n^3+9n^2+13n+6\right)$,

SO LHS = RHS IN $P(n+1)$ ✓ ■

2. SUPPOSE THAT $r \neq 1$.

TO PROVE THAT $\forall n \geq 0$, $\underbrace{1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: LHS OF $P(0) = 1$ $\quad$ ($r^0 = 1$, SO THE SUM HAS JUST ONE TERM), AND

RHS OF $P(0) = \frac{1 - r^{0+1}}{1 - r} = \frac{1 - r}{1 - r} = 1$, SO LHS = RHS ✓

CLAIM: $\forall n \geq 0$, $P(n) \Rightarrow P(n+1)$ → I.E., $\underset{\text{(LHS)}}{1 + r + r^2 + \cdots + r^n + r^{n+1}} = \underset{\text{(RHS)}}{\frac{1 - r^{n+2}}{1 - r}}$ Φ

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ $\quad (*)$

THEN $(1 + r + r^2 + \cdots + r^n) + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1}$ BY $(*)$

LHS

$$= \frac{1 - r^{n+1}}{1 - r} + \frac{(1 - r) r^{n+1}}{1 - r} = \frac{1 - \cancel{r^{n+1}} + \cancel{r^{n+1}} - r^{n+2}}{1 - r}$$

$$= \frac{1 - r^{n+2}}{1 - r} \checkmark \blacksquare$$

→ RHS

3. (a) TO PROVE $\forall n \geq 10$, $\underbrace{100n < 2^n}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(10)$

PROOF: LHS OF $P(10) = 100 \cdot 10 = 1000$, AND
RHS OF $P(10) = 2^{10} = 1024$, SO LHS < RHS ✓

CLAIM: $\forall n \geq 10$, $P(n) \Rightarrow P(n+1)$ → I.E., $\underset{\text{(LHS)}}{100(n+1)} < \underset{\text{(RHS)}}{2^{n+1}}$ Φ

PROOF: LET $n \geq 10$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $100n < 2^n$ $\quad (*)$

$100(n+1) = (100n) + 100 < 2^n + 100$ BY $(*)$

LHS

$< 2^n + 2^n$ SINCE $n \geq 10$

$= 2 \cdot 2^n = 2^{n+1}$, I.E., $100(n+1) < 2^{n+1}$ ✓ $\blacksquare$

(b) TO PROVE $\forall n \geq 4$, $\underbrace{2^n < n!}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(4)$

PROOF: LHS OF $P(4) = 2^4 = 16$, AND
RHS OF $P(4) = 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, SO LHS < RHS ✓

CLAIM: $\forall n \geq 4$, $P(n) \Rightarrow P(n+1)$ → I.E., $\underset{(LHS)}{2^{n+1}} < \underset{(RHS)}{(n+1)!}$

PROOF: LET $n \geq 4$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $2^n < n!$ (*)

THEN $2^{n+1} = 2 \cdot (2^n) < 2 \cdot n!$ BY (*)
LHS

AND $(n+1)! = (1 \cdot 2 \cdot 3 \cdots n)(n+1) = n!\,(n+1)$,
RHS

SO LHS < RHS BECAUSE $2 < (n+1)$. ✓ ■

(c) SUPPOSE $x > -1$. → LOOK CAREFULLY TO TRY TO FIND WHERE THIS IS NEEDED!

TO PROVE $\forall n \geq 0$, $\underbrace{(1+x)^n \geq 1 + nx}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: LHS OF $P(0) = (1+x)^0 = 1$, AND
RHS OF $P(0) = 1 + 0 \cdot x = 1$, SO LHS ≤ RHS ✓

CLAIM: $\forall n \geq 0$, $P(n) \Rightarrow P(n+1)$ → I.E., $(1+x)^{n+1} \geq 1 + (n+1)x$

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $(1+x)^n \geq 1 + nx$ (*)

THEN $(1+x)^{n+1} = (1+x) \cdot ((1+x)^n) \geq (1+x)(1+nx)$, BY (*)

WHICH $= 1 + \underbrace{x + nx}_{(n+1)x} + nx^2 = 1 + (n+1)x + nx^2$

$\geq 1 + (n+1)x$ SINCE $nx^2 \geq 0$. ✓ ■
↳ RHS

4. $0, \ 1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ 34, \ 55$

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
DEF.	DEF.	$F_2=F_1+F_0$	$F_3=F_2+F_1$	$F_4=F_3+F_2$	$F_5=F_4+F_3$	$F_6=F_5+F_4$	$F_7=F_6+F_5$	$F_8=F_7+F_6$	$F_9=F_8+F_7$	$F_{10}=F_9+F_8$

$(F_{n+1} = F_n + F_{n-1})$

5. (a) TO PROVE $\forall n \geq 0, \ \underbrace{F_0 + F_1 + \dots + F_n = F_{n+2} - 1}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: LHS OF $P(0) = F_0 = 0$, AND
RHS OF $P(0) = F_{0+2} - 1 = F_2 - 1 = 1 - 1 = 0$, SO LHS = RHS ✓

CLAIM: $\forall n \geq 0, \ P(n) \Rightarrow P(n+1)$ → I.E., $\underbrace{F_0 + F_1 + \dots + F_n + F_{n+1}}_{(LHS)} \overset{?}{=} \underbrace{F_{n+3} - 1}_{(RHS)}$

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $F_0 + F_1 + \dots + F_n = F_{n+2} - 1$ $(*)$

$\underbrace{(F_0 + F_1 + \dots + F_n) + F_{n+1}}_{(LHS)} = (F_{n+2} - 1) + F_{n+1}$ BY $(*)$

$= (F_{n+2} + F_{n+1}) - 1$

$= \underbrace{F_{n+3} - 1}_{(RHS)}$ ✓ ∎

(b) TO SHOW THAT $\forall n \geq 0, \ \underbrace{F_0^2 + F_1^2 + \dots + F_n^2 = F_n F_{n+1}}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: LHS OF $P(0) = F_0^2 = 0^2 = 0$, AND
RHS OF $P(0) = F_0 F_{0+1} = F_0 F_1 = 0 \cdot 1 = 0$, SO LHS = RHS ✓

CLAIM: $\forall n \geq 0, \ P(n) \Rightarrow P(n+1)$ → I.E., $\underbrace{F_0^2 + F_1^2 + \dots + F_n^2 + F_{n+1}^2}_{(LHS)} \overset{?}{=} \underbrace{F_{n+1} F_{n+2}}_{(RHS)}$

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $F_0^2 + F_1^2 + \dots + F_n^2 = F_n F_{n+1}$ $(*)$

$\underbrace{(F_0^2 + F_1^2 + \dots + F_n^2) + F_{n+1}^2}_{(LHS)} = F_n F_{n+1} + F_{n+1}^2$ BY $(*)$

$= F_{n+1}(F_n + F_{n+1}) = \underbrace{F_{n+1} F_{n+2}}_{(RHS)}$ ✓ ∎

(c) TO SHOW $\forall n \geq 0$, $\underbrace{\left[F_{2n+1} = F_{n+1}^2 + F_n^2\right] \wedge \left[F_{2n+2} = F_{n+2}^2 - F_n^2\right]}_{P(n)}$

BY INDUCTION ON n:

CLAIM: $P(0)$

PROOF: $P(0) \iff \left[F_1 = F_1^2 + F_0^2\right] \wedge \left[F_2 = F_2^2 - F_0^2\right]$

$\iff \left[1 = 1^2 + 0^2\right] \wedge \left[1 = 1^2 - 0^2\right]$

$\iff$ TRUE $\wedge$ TRUE $\iff$ TRUE ✓

CLAIM: $\forall n \geq 0$, $P(n) \Rightarrow P(n+1)$ I.E., $\left[\underset{(a)}{F_{2n+3}} = \underset{(b)}{F_{n+2}^2 + F_{n+1}^2}\right] \wedge \left[\underset{(c)}{F_{2n+4}} = \underset{(d)}{F_{n+3}^2 - F_{n+1}^2}\right]$

PROOF: LET $n \geq 0$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $\left[\underset{(*)}{F_{2n+1} = F_{n+1}^2 + F_n^2}\right] \wedge \left[\underset{(**)}{F_{2n+2} = F_{n+2}^2 - F_n^2}\right]$.

THEN $\underset{(a)}{F_{2n+3}} = F_{2n+2} + F_{2n+1} \overset{(**),\,(*)}{=} \left(F_{n+2}^2 - \cancel{F_n^2}\right) + \left(F_{n+1}^2 + \cancel{F_n^2}\right) = \underset{(b)}{F_{n+2}^2 + F_{n+1}^2}$ ✓

AND $\underset{(c)}{F_{2n+4}} = F_{2n+3} + F_{2n+2} \overset{(a)=(b),\,(**)}{=} \left(F_{n+2}^2 + F_{n+1}^2\right) + \left(F_{n+2}^2 - F_n^2\right)$

(OK, NO MORE $2n$'S, LET'S GET IT DOWN TO JUST F_{n+1} & F_n)

$= 2F_{n+2}^2 + F_{n+1}^2 - F_n^2 = 2\left(F_{n+1} + F_n\right)^2 + F_{n+1}^2 - F_n^2$

$= 2\left(F_{n+1}^2 + 2F_{n+1}F_n + F_n^2\right) + F_{n+1}^2 - F_n^2$

$= 2F_{n+1}^2 + 4F_{n+1}F_n + 2F_n^2 + F_{n+1}^2 - F_n^2$

$= 3F_{n+1}^2 + 4F_{n+1}F_n + F_n^2$

$\underset{(d)}{F_{n+3}^2 - F_{n+1}^2} = \left(F_{n+2} + F_{n+1}\right)^2 - F_{n+1}^2 = F_{n+2}^2 + 2F_{n+2}F_{n+1} + \cancel{F_{n+1}^2} - \cancel{F_{n+1}^2}$

$= \left(F_{n+1} + F_n\right)^2 + 2\left(F_{n+1} + F_n\right)F_{n+1}$

$= F_{n+1}^2 + 2F_{n+1}F_n + F_n^2 + 2F_{n+1}^2 + 2F_{n+1}F_n$

$= 3F_{n+1}^2 + 4F_{n+1}F_n + F_n^2$

$\therefore$ (a) = (b) $\wedge$ (c) = (d) ✓ ■

(d) TO SHOW $\forall n \geq 1$, $\underbrace{F_{n+1} F_{n-1} = F_n^2 + (-1)^n}_{P(n)}$ BY INDUCTION ON n:

CLAIM: $P(1)$

PROOF: LHS OF $P(1) = F_2 \cdot F_0 = 1 \cdot 0 = 0$, AND

RHS OF $P(1) = F_1^2 - 1 = 1^2 - 1 = 0$, SO LHS = RHS ✓

CLAIM: $\forall n \geq 1$, $P(n) \Rightarrow P(n+1)$ → I.E., $\underbrace{F_{n+2} F_n}_{(LHS)} \overset{?}{=} \underbrace{F_{n+1}^2 + (-1)^{n+1}}_{(RHS)}$

PROOF: LET $n \geq 1$ BE GIVEN.

SUPPOSE $P(n)$, I.E., $F_{n+1} F_{n-1} = F_n^2 + (-1)^n$ (*) → NOTE $F_n^2 = F_{n+1} F_{n-1} - (-1)^n$

$= F_{n+1} F_{n-1} + (-1)^{n+1}$

THEN $\underbrace{F_{n+2} F_n}_{(LHS)} = (F_{n+1} + F_n) F_n = F_{n+1} F_n + (F_n^2)$

$= F_{n+1} F_n + \left(F_{n+1} F_{n-1} + (-1)^{n+1}\right)$ BY (*)

$= F_{n+1} (F_n + F_{n-1}) + (-1)^{n+1}$

$= F_{n+1} \cdot F_{n+1} + (-1)^{n+1}$

$= \underbrace{F_{n+1}^2 + (-1)^{n+1}}_{(RHS)}$ ✓ ∎

6. RECALL THE FORMULAE:

$\boxed{F_{odd}}$ $\quad F_{2n+1} = F_{n+1}^2 + F_n^2$ $\qquad \boxed{F_{even}}$ $\quad F_{2n+2} = F_{n+2}^2 - F_n^2$

CURIOUS NOTE: THE INDICES ADD UP!
$2n+1 = (n+1)+n$ & $2n+2 = (n+2)+n$

(a) PATTERN-MATCHING, $\underbrace{F_9}_{n=4} = F_5^2 + F_4^2 = 5^2 + 3^2 = 25 + 9 = 34$ ✓

AND $\underbrace{F_{10}}_{n=4} = F_6^2 - F_4^2 = 8^2 - 3^2 = 64 - 9 = 55$ ✓

(b) AGAIN PATTERN, MATCHING: $\underbrace{F_{15}}_{n=7} = F_8^2 + F_7^2 = 21^2 - 13^2 = 441 + 169 = \underline{610}$;

$\underbrace{F_{16}}_{n=7} = F_9^2 - F_7^2 = 34^2 - 13^2 = 1156 - 169 = \underline{987}$; AND

$\underbrace{F_{17}}_{n=8} = F_9^2 + F_8^2 = 1156 + 441 = \underline{1597}$

(c) ONE MORE TIME: $\underbrace{F_{31}}_{n=15} = F_{16}^2 + F_{15}^2 = 987^2 + 610^2 = \underline{1346269}$

$\underbrace{F_{32}}_{n=15} = F_{17}^2 - F_{15}^2 = 1597^2 - 610^2 = \underline{2178309}$ → JENNY?

$\underbrace{F_{33}}_{n=16} = F_{17}^2 + F_{16}^2 = 1597^2 + 987^2 = \underline{3524578}$

(d) $\underline{F_{17} = F_{16} + F_{15}}$, AND $\underline{F_{33} = F_{32} + F_{31}}$, WHICH YOU CAN SEE IN THE SUMS & DIFFERENCES OF SQUARES!